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ON CALABI-YAU THREEFOLDS WITH INFINITE FUNDAMENTAL GROUP

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Apology from the first author.

First of all, I should apologise the change of the title of this report. At the Kinosaki Conference 1998 I have presented some part of my joint work with D. Q. Zhang about finite automorphism groups of K3 surfaces. However, around the last December, I have started and gradually concentrated to study another subject, Calabi-Yau threefolds with infinite fundamental group, jointly with my student, Jun Sakurai. Till now I have been occupied more or less by this subject. This is completely elementary but turns out to be more interesting than I expected before starting and we have now obtained some results which, I hope, are worth being reported here.

Introduction.

Throughout this paper, we call a smooth compact Kählerian threefold X a Calabi-Yau threefold if it satisfies $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$. This definition of Calabi-Yau threefold is adopted by many algebraic geometers and is indeed parallel to that of K3 surface. However, contrary to the case of K3 surface, these two conditions $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$ for a Kählerian threefold X imply, on one hand, $h^2(\mathcal{O}_X) = 0$ whence the projectivity of X [Kd], but, on the other hand, neither the simply-connectedness of X even nor the finiteness of $\pi_1(X)$ as is illustrated by the following two examples:

Example 1 (Igusa's example; [I, Page 678], [U, Example 16.16]). *Let E_i $(i = 1, 2, 3)$ be three elliptic curves with origin 0 and $P_i \in (E_i)_2 - \{0\}$ a non-zero two torsion point of E_i. Consider the abelian threefold $A := E_1 \times E_2 \times E_3$ and its involutions, $g := t_{(P_1,0,0)} \circ (id_{E_1} \times -id_{E_2} \times -id_{E_3})$ and $h := t_{(0,P_2,P_3)} \circ (-id_{E_1} \times -id_{E_2} \times id_{E_3})$, where t_* stands for the translation by an element $*$ of A. Then $\langle g, h\rangle = \{id, g, h, gh\} (\simeq C_2^{\oplus 2})$ acts freely on A and the quotient variety $X := A/\langle g, h\rangle$ is a Calabi-Yau threefold whose fundamental group $\pi_1(X)$ fits in with the exact sequence $0 \to \mathbb{Z}^{\oplus 6} \to \pi_1(X) \to C_2^{\oplus 2} \to 0$.* □

Example 2 (eg. [Og1, Example 3.2]). *Let E be an elliptic curve, T an Enriques surface, $\pi : S \to T$ the universal covering of T and ι the covering involution of $\pi : S \to T$. Consider the product $S \times E$ and its involution $g := \iota \times -id_E$. Then $\langle g\rangle (\simeq C_2)$ acts freely on $S \times E$ and the quotient variety $X := (S \times E)/\langle g\rangle$ is a Calabi-Yau threefold whose fundamental group $\pi_1(X)$ fits in with*

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX *the exact sequence* $0 \to \mathbb{Z}^{\oplus 2} \to \pi_1(X) \to C_2 \to 0$. *Note also that the projection* $p_1 : S \times E \to S$ *induces an elliptic fibration* $\overline{p}_1 : X \to T$. □

Example 1 also provides an explicit example of a Calabi-Yau threefold which contains no rational curves, while there is a conjecture which predicts that every Calabi-Yau threefold with finite fundamental group contains a rational curve (cf. [Mo]). Example 2 shows that there exists a Calabi-Yau threefold admitting an elliptic fibration whose base space is not a rational surface, while it can be shown that the base space of an elliptically fibered Calabi-Yau threefold is rational whenever $\pi_1(X)$ is finite (cf. [Og1 Lemma 3.4]). So, in the study of Calabi-Yau threefolds, it is sometimes inevitable to distinguish the case where $\pi_1(X)$ is infinite.

The goal of this paper is to describe all the possible infinite fundamental groups of Calabi-Yau threefolds in the form of group extension (Corollary 1) and to get a fairly practical criterion for Calabi-Yau threefolds to have finite fundamental group in terms of Picard number and the one for those to have non-trivial second Chern class $c_2(X)$ (Corollary 2). Consult [W2] and [O3,4] for the importance of the role of c_2 in Calabi-Yau classifications.

Let X be a Calabi-Yau threefold with infinite fundamental group. Acoording to the Bogomolov decomposition Theorem ([Be1, 2]), such an X admits an etale Galois covering from either an abelian threefold or the product of a K3 surface and an elliptic curve. We call X of Type A

in the former case and of Type K in the latter case. Among many candidates of such coverings for a given X, we always fix the smallest one called the minimal splitting cover, which we can always obtain by posing one additional condition on the Galois group G that G contains no non-zero translations in the case where X is of Type A and that G contains no elements of the form $(id_S, \text{non-zero translation of} E)$ in the case where X is of Type K ([Be2, Section 3], see also Definitions (1.1) and (2.1)).

Adopting this convention and using notation listed at the end of Introduction, we can state our main result as follows:

Main Theorem.
[1] Let X be a Calabi-Yau threefold of Type A and G the Galois group of the minimal splitting covering. Then,

(1) *G is isomorphic to either $C_2^{\oplus 2}$ or D_8;*
(2) *Conversely, each of these two groups occurs as the Galois group of the minimal splitting cover of some Calabi-Yau threefold of Type A;*
(3) *In each case of [1](1), the Piacrd number $\rho(X)$ of X, which is equal to $h^1(T_X)$ the dimension of the Kuranishi space of X, is determined uniquely by G and is calculated as in the following table:*

G	$C_2^{\oplus 2}$	D_8
$\rho(X)$	3	2

[2] Let X be a Calabi-Yau threefold of Type K and G the Galois group of

the minimal splitting covering. Then,

(1) *G is isomorphic to either $C_2^{\oplus n}$ $(1 \leq n \leq 3)$, D_{2n} $(3 \leq n \leq 6)$ or $C_3^{\oplus 2} \rtimes C_2$;*
(2) *Conversely, each of these groups occurs as the Galois group of the minimal splitting cover of some Calabi-Yau threefold of Type K except possibly for D_{2n} $(3 \leq n \leq 6)$ and $C_3^{\oplus 2} \rtimes C_2$;*
(3) *In each case of [2](1), $\rho(X)$, which is again equal to $h^1(T_X)$, is determined uniquely by G and is calculated as in the following table:*

G	C_2	$C_2^{\oplus 2}$	$C_2^{\oplus 3}$	D_6	D_8	D_{10}	D_{12}	$C_3^{\oplus 2} \rtimes C_2$
$\rho(X)$	11	7	5	5	4	3	3	3

□

As an immediate Corollary, we get the following:

Corollary 1. *Let X be a Calabi-Yau threefold and assume that $\pi_1(X)$ is infinite. Then $\pi_1(X)$ falls into one of the following exact sequences:*
$0 \to \mathbb{Z}^{\oplus 6} \to \pi_1(X) \to G \to 1$, *where G is isomorphic to either $C_2^{\oplus 2}$ or D_8;*
$0 \to \mathbb{Z}^{\oplus 2} \to \pi_1(X) \to G \to 1$, *where G is isomorphic to either $C_2^{\oplus n}$ $(1 \leq n \leq 3)$, D_{2n} $(3 \leq n \leq 6)$ or $C_3^{\oplus 2} \rtimes C_2$. In particular, $\pi_1(X)$ is always solvable.* □

Taking the contraposition of the main Theorem, we also obtain the next:

Corollary 2. *Let X be a Calabi-Yau threefold. Then,*

(1) *If $\rho(X) = 1, 6, 8, 9, 10$ or greater than or equal to 12, then $\pi_1(X)$ is finite.*
(2) *If $\rho(X) = 1$ or $\rho(X) \geq 4$, then X is not an etale quotient of an abelian threefold. In particular, the second Chern class $c_2(X) \neq 0$ in $H^4(X, \mathbb{R})$.* □

The last statement follows from the fact that a Kählerian manifold X such that $c_1(X) = c_2(X) = 0$ in $H^*(X, \mathbb{R})$ is an etale quotient of a complex torus [Kb, Chap.IV, Corollary (4.15)].

Except some concrete examples [Be1], [F], [Be3], very little are known about finite, non-trivial fundamental groups of Calabi-Yau threefolds and it is a little bit surprising for the authors that the fundamental groups of Calabi-Yau threefolds with Picard number one are always finite, while they should confess at the same time that they do not know whether there actually exists a Calabi-Yau threefold such that $\rho(X) = 1$ and $c_3(X) = 0$, i.e., $\rho(X) = h^1(T_X) = 1$. For the last phrase, note that $\pi_1(X)$ is finite whenever $c_3(X) \neq 0$ by the Bogomolov decomposition Theorem.

Apart from its own interest, Corollary 1 (2) together with the main Theorem [1] and Wilson's insight [W1, Page 141], "The author should confess however to his feeling that by using more specific information on the cup product, one might hope for a result along this line that any Calabi-Yau manifold (threefold) is the resolution of a Calabi-Yau model with $\rho \leq 3$.", leads us to the following:

Question.

(1) *Does any Calabi-Yau threefold X such that $\rho(X) \geq 4$ admit a non-trivial birational contraction? (The affirmative answer implies the existence of a rational curve on X [Ka, Theorem 1]).*

(2) *Does any Calabi-Yau threefold whose Picard number is one contain a rational curve?* □

Concerning Question (1), the best result known to the authors is that of D.R. Heath-Brown and P.M.H. Wilson which asserts that X admits a non-trivial birational contraction whenever $\rho(X) \geq 14$ [HW]. It is also well known that Question (2) is affirmative for a complete intersection Calabi-Yau threefold. Indeed, such a Calabi-Yau threefold always contains several lines and more (eg. [EJS]).

This paper has grown out of the second author's master thesis at University of Tokyo 1998 under the first author's instruction. Both authors would like to express their gratitude to Professor Y. Kawamata for his warm encouragement.

Notation and Convention.

Throughout this paper, we employ the following notation and convention:
$\zeta_n := \exp(2\pi\sqrt{-1}/n)$, the primitive n-th root of unity in $\mathbb{C}$;
$C_n := \langle a | a^n = 1 \rangle$, the cyclic group of order n;
$D_{2n} := \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle \simeq C_n \rtimes C_2$, the dihedral group of order $2n$;
$C_3^{\oplus 2} \rtimes C_2$, the semi-direct product of $C_3^{\oplus 2}$ and $C_2 := \langle \iota \rangle$ whose semi-dirct product structure is given by $\iota h \iota = h^{-1}$ for each $h \in C_3^{\oplus 2}$;
$Q_{4n} := \langle a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$, the binary dihedral group of order $4n$;
$S_n := \mathrm{Aut}_{\mathrm{set}}(\{1, 2, \cdots, n\})$, the n-th symmetric group;
$A_n := \mathrm{Ker}(\mathrm{sgn} : S_n \to \{\pm 1\})$, the n-th alternative group;
T_{24}, O_{48}, I_{120}, the binary tetrahedral group of order 24, the binary octahedral group of order 48, and the binary isosahedral group of order 120 (for the precise definition, see for example [YY, Pages 12-14]);
When a group G acts on a set S, we need to distinguish several sets of the fixed points by:
$S^g := \{s \in S | g(s) = s\}$ for $g \in G$;
$S^G := \cap_{g \in G} S^g$, the set of points which are fixed by all the elements of G;
$S^{[G]} := \cup_{g \in G - \{1\}} S^g$, the set of points which are fixed by some non-trivial element of G. Note that the action of G on S is said to be fixed point free if $S^{[G]} = \emptyset$.
For a d-dimensional smooth complete variety X with $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X$, we denote by ω_X a nowhere vanishing holomorphic d-form on X.
Let $A := \mathbb{C}^d/\Lambda$ be a d-dimensional complex torus. By abuse of language, we call global coordinates $(z_1, z_2, ..., z_d)$ of $\mathbb{C}^n$ global coordinates of A if they are obtained by an affine transformation of the natural global coordinates of $\mathbb{C}^d$ given by the i-th projections. When the origin 0 of A is chosen and A is regarded as a group variety, we identify A with its translation group in a natural manner and denote by $(A)_n$ the group of n-torsion points.
In this paper, we often regard group actions on varieties as the so-called co-action through their coordinates. The advantage of this lies in the fact that any actions on cohomology groups are then described in a covariant way like $(ab)^* = a^*b^*$.

§1. Calabi-Yau threefolds of Type A.

In this section we study Calabi-Yau threefolds of Type A. Let us define:

Definition (1.1). *We call a finite group G a Calabi-Yau group of Type A (resp. a pre-Calabi-Yau group of Type A) if there exist an abelian threefold A and a faithful* *representation $G \hookrightarrow Aut(A)$ which satisfy the following conditions (1) - (4) (resp. (1) - (3)):*

(1) *G contains no non-zero translations;*
(2) *$g^*\omega_A = \omega_A$ for all $g \in G$;*
(3) *$A^{[G]} = \emptyset$;*
(4) *$H^0(A, \Omega_A^1)^G = \{0\}$.* $\square$

Throughout this section, we abbreviate a Calabi-Yau group of Type A and a pre-Calabi-Yau group of Type A simply by a C.Y. group and a pre-C.Y. group. The goal of this section is to prove the following:

Theorem (1.2). *Let G be a C.Y. group and A a target abelian threefold. Then,*

(1) *G is isomorphic to either $C_2^{\oplus 2}$ or D_8. Conversely, each of these two groups is a C.Y. group.*
(2) *Moreover, there exists a basis of $H^0(A, \Omega_A^1)$ under which the natural representaion of G on $H^0(A, \Omega_A^1)$ is described as follows:*
If $G = \langle a, b \rangle \simeq C_2^{\oplus 2}$, then
$$a^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } b^* = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
If $G = \langle a, b | a^4 = b^2 = 1, bab = a^{-1} \rangle \simeq D_8$, then
$$a^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } b^* = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad \square$$

We observe first that Theorem (1.2) implies the main Theorem [1].

Proof of the main Theorem [1] assuming Theorem (1.2).

Let X be a Calabi-Yau threefold of Type A. Choose an abelian threefold A and an etale Galois covering $\pi : A \to X$. Denote by G the Galois group of π. Fix an origin of A and set $H := G \cap A$. Then H is a normal subgroup of G. Moreover, the induced action of G/H on the abelian threefold A/H satisfies the conditions (1)-(4) in (1.1) and keeps the property $(A/H)/(G/H) = X$. So, replacing (G, A) by $(G/H, A/H)$, we may assume from the first that G itself is a C.Y. group and A its target abelian threefold. Then, by (1.2)(1), G is isomorphic to either $C_2^{\oplus 2}$ or D_8. This proves the assertion [1](1) of the main Theorem. Let us fix global coordinates (z_1, z_2, z_3) of A such that $\langle dz_1, dz_2, dz_3 \rangle$ gives a basis of $H^0(A, \Omega_A^1)$ found in (1.2)(2). Recall that $H^2(A, \mathbb{C}) = \wedge^2 H^1(A, \mathbb{C})$ and that $H^1(A, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^0(A, \Omega_A^1) \oplus \overline{H^0(A, \Omega_A^1)}$ under the identification $H^*(A, \mathbb{C}) = H^*_{\mathrm{DR}}(A, \mathbb{C})$. Using these equalities and the description given in (1.2)(2), we readily find that $H^2(A, \mathbb{C})^G = \mathbb{C}\langle dz_1 \wedge d\bar{z}_1, dz_2 \wedge d\bar{z}_2, dz_3 \wedge d\bar{z}_3 \rangle$ when $G \simeq C_2^{\oplus 2}$ and $H^2(A, \mathbb{C})^G = \mathbb{C}\langle dz_1 \wedge d\bar{z}_1, dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3 \rangle$ when $G \simeq C_2^{\oplus 2}$. Combining this with $H^*(X, \mathbb{C}) \simeq H^*(A, \mathbb{C})^G$, we get $B_2(X) = 3$ when $G \simeq C_2^{\oplus 2}$ and $B_2(X) = 2$ when $G \simeq D_8$. Recall the formulae $h^0(\Omega_X^2) = h^2(\mathcal{O}_X) = 0$, $B_2(X) = h^{1,1}(X) = \rho(X)$ and $c_3(X) = 2(h^{1,1}(X) - h^{2,1}(X)) = 2(h^{1,1}(X) - h^1(T_X))$ for a Calabi-Yau threefold. Now combining these equilities with $c_3(X) = c_3(A)/|G| = 0$ and $H^2(X, \mathbb{C}) \simeq H^2(A, \mathbb{C})^G$, we obtain the formulae in the main Theorem [1](3). Conversely, if G is a C.Y. group whose target abelian threefold is A, then the quotient space A/G is a Calabi-Yau threefold by (1.1) and by the Hodge symmetry $h^1(\mathcal{O}_X) = h^0(\Omega^1_X)$. This together with the last statement of (1.2)(1) implies the assertion [1](2) of the main Theorem. $\square$

Next, we observe that $C_2^{\oplus 2}$ and D_8 are actually C.Y. groups.

Proof of the fact that $C_2^{\oplus 2}$ and D_8 are C.Y. groups.

We already observed by Igusa's example (Example 1 in Introduction) that $C_2^{\oplus 2}$ is a C.Y. group. So it is enough to construct an abelian threefold on which D_8 acts as a C.Y. group. Let us first take two elliptic curves E_1 and E_2 and consider the product abelian threefold $\tilde{A} = E_1 \times E_2 \times E_2$. Let us fix points $\tau_1 \in (E_1)_4 - (E_1)_2$, $\tau_2, \tau_3 \in (E_2)_2$ such that $\tau_2 \neq \tau_3$ and define automorphisms $\tilde{a}$ and $\tilde{b}$ of $\tilde{A}$ by $\tilde{a}(z_1, z_2, z_3) = (z_1 + \tau_1, -z_3, z_2)$ and $\tilde{b}(z_1, z_2, z_3) = (-z_1, z_2 + \tau_2, -z_3 + \tau_3)$. Set $\tilde{G} = \langle \tilde{a}, \tilde{b} \rangle$. Then we readily calculate that $\tilde{a}^4 = \tilde{b}^2 = id$, $\tilde{a}\tilde{b}\tilde{a}\tilde{b} = t_\tau$, $\tilde{a}t_\tau\tilde{a}^{-1} = t_\tau$ and $\tilde{b}t_\tau\tilde{b}^{-1} = t_\tau$, where $\tau = (0, \tau_2 + \tau_3, \tau_2 + \tau_3)$. In particular, $\langle t_\tau \rangle (\simeq C_2)$ is a normal subgroup of $\tilde{G}$. Set $A := \tilde{A}/\langle t_\tau \rangle$, $G := \tilde{G}/\langle t_\tau \rangle$, $a := \tilde{a} \bmod \langle t_\tau \rangle$, and $b := \tilde{b} \bmod \langle t_\tau \rangle$. Then $G = \langle a, b \rangle$ and G acts on A in a natural manner. We show that this pair (G, A) gives a desired example. By the definition of a and b, we have $a^*|H^0(A, \Omega^1_A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ and $b^*|H^0(A, \Omega^1_A) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ under the basis $\langle dz_1, dz_2, dz_3 \rangle$. It follows from these descriptions that $a^*\omega_A = \omega_A$, $b^*\omega_A = \omega_A$ and that the image of the natural representaion $\rho : G \to \mathrm{GL}(H^0(A, \Omega^1_A))$ is isomorphic to D_8. Combining this with the relations $a^4 = b^2 = id$ and $bab = a^{-1}$ derived from those for $\tilde{a}$ and $\tilde{b}$, we find that $G \simeq \mathrm{Im}(\rho) \simeq D_8$. This, in particular, implies that G contains no non-trivial translations.

Now it remains to show that $A^{[G]} = \emptyset$. For this it is sufficient to check that for each $\tilde{c} \in \{\tilde{a}^i\tilde{b}^j | 0 \leq i \leq 3, 0 \leq j \leq 1\} - \{id, t_\tau\}$, there are no (z_1, z_2, z_3) such that either $\tilde{c}(z_1, z_2, z_3) = (z_1, z_2, z_3)$ or $\tilde{c}(z_1, z_2, z_3) = (z_1, z_2 + \tau_2 + \tau_3, z_3 + \tau_2 + \tau_3)$. However, using $\tau_1, 2\tau_1, 3\tau_1 \neq 0$ and $\tau_2, \tau_3, \tau_2 + \tau_3 \neq 0$, we can readily check this by an explicit calculation. $\square$

It remains to prove the first part of (1.2)(1) and the assertion (1.2)(2). For this, it is more convenient to consider not only C.Y. groups but also pre-C.Y.groups because of the following inductive property:

Lemma (1.3). *If G is a pre-C.Y. group, then so are all the subgroups of G. In other word, if a finite group G contains a subgroup which is not a pre-C.Y. group, then G itself is not a pre-C.Y. group.*

Proof. It follows from the fact that the conditions (1)-(3) in (1.1) is closed under taking a subgroup. $\square$

Lemma (1.4). *Let G be a pre-C.Y. group, A its target abelian threefold and $\rho : G \to GL(H^0(A, \Omega^1_A)); g \mapsto g^*$ the natural representation. Then,*

(1) *ρ is injective;*
(2) *$Im(\rho) \subset SL(H^0(A, \Omega^1_A))$;*

(3) *Let g be an element of G of order n. Then, there exists a basis of $H^0(A, \Omega^1_A)$ depending on g under which the matrix representaion of $g^*|H^0(A, \Omega^1_A)$ is of the form $diag(1, \zeta_n, \zeta_n^{-1})$. Moerover, $n \in \{1, 2, 3, 4, 6\}$.*

Proof. The assertion (1) follows from (1.1)(1). The assertion (2) follows from (1.1)(3), because $H^0(A, \Omega^3_A) = \wedge^3 H^0(A, \Omega^1_A)$. Let us show the assertion (3). First fix a basis of $H^0(A, \Omega^1_A)$ under which the matrix representaion of g^* is of the form $\mathrm{diag}(a, b, c)$. Suppose $a \neq 1$, $b \neq 1$, and $c \neq 1$. Then, there exist global coordinates (x, y, z) of A and complex numbers p, q, r such that the (co-)action of g on A is written as $g(x, y, z) = (ax + p, by + q, cz + r)$. However, the point $P = (p/(1-a), q/(1-b), r/(1-c)) \in A$ is then a fixed point of g, a contradiction. This shows the first half part of the assertion (3). The middle part of (3) is now clear by (1) and (2). Let us show the last part of the assertion (3). Since $H^1(A, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^0(A, \Omega^1_A) \oplus \overline{H^0(A, \Omega^1_A)}$, the matrix representation g on $H^1(A, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ is of the form $g^* = \mathrm{diag}(1, \zeta_n, \zeta_n^{-1}, 1, \overline{\zeta_n}, \overline{\zeta_n^{-1}}) = \mathrm{diag}(1, \zeta_n, \zeta_n^{-1}, 1, \zeta_n^{-1}, \zeta_n)$. This together with the fact that g^* is at the same time defined over $H^1(A, \mathbb{Z})$ implies $\varphi(n) \leq (6-2)/2 = 2$, whence $n \in \{1, 2, 3, 4, 6\}$. □

Next we determine commutative pre-C.Y. groups.

Lemma (1.5). *Let G be a commutative pre-C.Y. group and A a target abelain threefold. Then G is isomorphic to either one of C_n $(1 \leq n \leq 6,\ n \neq 5)$ or $C_2^{\oplus 2}$. In particular, there exist no commutative pre-C.Y. groups of order greater than or equal to 7. Moreover, if G is a commutative C.Y. group, then G is isomorphic to $C_2^{\oplus 2}$ and the action of G on $H^0(A, \Omega^1_A)$ is same as in (1.2) (3).*

Proof. By the structure Theorem of finite abelian group (eg. [Kt, Chap.3, Theorem (3.4)]), we can write $G = \langle g_1 \rangle \oplus \cdots \oplus \langle g_r \rangle \simeq C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$, where $r \geq 0$ and $2 \leq n_1 | n_2 | \cdots | n_r$. In the case where $r \leq 1$, we get the result by (1.4)(3). Assume that $r \geq 2$. Let i, j be two integers such that $1 \leq i < j \leq r$. Using $g_i g_j = g_j g_i$ and (1.4)(3) and replacing g_i and g_j by other generaters of $\langle g_i \rangle$ and $\langle g_j \rangle$ if necessary, we may choose a basis of $H^0(A, \Omega^1_A)$ under which $g_i^*|H^0(A, \Omega^1_A)$ and $g_j^*|H^0(A, \Omega^1_A)$ are simulteneously diagonalised as either one of the following forms:
(1) $g_i^* = \mathrm{diag}(1, \zeta_{n_i}, \zeta_{n_i}^{-1})$ and $g_j^* = \mathrm{diag}(1, \zeta_{n_j}, \zeta_{n_j}^{-1})$ or
(2) $g_i^* = \mathrm{diag}(1, \zeta_{n_i}, \zeta_{n_i}^{-1})$ and $g_j^* = \mathrm{diag}(\zeta_{n_j}^{-1}, 1, \zeta_{n_j})$.

In the former case, we have $g_i^* = (g_j^*)^{n_j/n_i}$, whence $g_i = (g_j)^{n_j/n_i}$ by (1.4)(1), a contradiction. In the latter case, we calculate $g_i^* g_j^* = \mathrm{diag}(\zeta_{n_j}^{-1}, \zeta_{n_i}, \zeta_{n_i}^{-1}\zeta_{n_j})$ and $(g_i^{-1})^* g_i^* = \mathrm{diag}(\zeta_{n_j}^{-1}, \zeta_{n_i}^{-1}, \zeta_{n_i}\zeta_{n_j})$. Thus, by (1.4)(3), we find that $\zeta_{n_i}^{-1}\zeta_{n_j} = \zeta_{n_i}\zeta_{n_j} = 1$. This implies $\zeta_{n_i} = \zeta_{n_j} = -1$, whence $n_i = 2$ for all $i = 1, 2, \cdots, r$ by (1.4)(3). Assume that $r \geq 3$. Then, under an appropriate basis of $H^0(A, \Omega^1_A)$, we have $g_1^* = \mathrm{diag}(1, -1, -1)$ and $g_2^* = \mathrm{diag}(-1, 1, -1)$. Then g_3^* must be of the form $\mathrm{diag}(-1, -1, 1)$. However, this implies $g_1^* g_2^* = g_3^*$, whence $g_1 g_2 = g_3$ by (1.4)(1), a contradiction. Thus, $r = 2$, that is, $G \simeq C_2^{\oplus 2}$ and there exists a basis of $H^0(A, \Omega^1_A)$ under which $g_1^* = \mathrm{diag}(1, -1, -1)$ and $g_2^* = \mathrm{diag}(-1, 1, -1)$. From this description, we also find that $H^0(A, \Omega^1_A)^G = \{0\}$ if $G \simeq C_2^{\oplus 2}$. □

Next, we examine non-comutative pre-C.Y. groups. First we estimate their orders. For this, we make use of the following Theorems known in the Group Theory:

Theorem (1.6) (Wielandt, eg. [Kt, Chap.2, Theorem (2.2)]). *Let G be an arbitray finite group, p a prime number and a a positive integer such that $p^a||G|$. Then there exists a subgroup H of G such that $|H| = p^a$.*

Proof. This follows from the Sylow Theorem and the well-know property of $p-$group K that there exists a sequence of subgroups $\{1\} = K_0 < K_1 < K_2 < \cdots < K_{n-1} < K_n = K$ such that for all i, K_{i-1} is a normal subgroup of K_i and that $K_i/K_{i-1} \simeq C_p$. See for example [Kt, ibid.] for a direct proof. □

Theorem (1.7) (Burnside-Hall, eg. [Su, Page 90, Corollary 2]). *Let K be an arbitray $p-$group and H a maximal, normal commutative subgroup of G. Set $|G| = p^n$ and $|H| = p^a$. Then $a(a+1) \geq 2n$.* □

Remark. *This Theorem is also applied in [Mu] to study finite symplectic automorphism groups of K3 surfaces.* □

Let us return back to pre-C.Y. groups.

Lemma (1.8). *Let G be a pre-C.Y. group. Then, $|G|$ is either 2^n or $2^n \cdot 3$, where n is an integer such that $0 \leq n \leq 3$. In orther words, $|G|$ is either one of $1, 2, 3, 4, 6, 8, 12$ or 24.*

Proof. By (1.6) and (1.4)(3), there exist non-negative integers m and n such that $|G| = 2^n \cdot 3^m$. Assume that $m \geq 2$. Then, it follows from (1.6) that G contains a subgroup H of order 3^2. Then H is a pre-C.Y. group of Type A (1.3) and is isomorphic to either $C_3^{\oplus 2}$ or C_9. However, this contradicts (1.5). Thus $m = 0$ or 1. Next assume that $n \geq 4$. Then by (1.6), G contains a subgroup H of order 2^4. Let K be a maximal normal commutative subgroup of H and set $|K| = 2^a$. Then, applying (1.7), we find that $a \geq 3$. However, this again contradicts (1.5) and (1.3). Thus $n \leq 3$. □

Combining this with the classification of non-commutative finite groups of small order (eg. [Bu, Chap.4, Pages 54-55 and Chap.5, Pages 83-89]), we get:

Corollary (1.9). *Let G be a pre-C.Y. group. Assume that G is non-commutative and that $|G| \leq 12$. Then G is isomorphic to either one of $D_6 (\simeq S_3)$, D_8, Q_8, D_{12}, Q_{12} or A_4.* □

We show that among the candidates in (1.9), only D_8 is realised as a C.Y. group of Type A. For this, we make use of the following:

Proposition (1.10) (eg. [Kt, Chap.8, Pages 273-275]). *Up to isomorphisms, the complex linear irreducible representaions of D_{2n} $(3 \leq n \in \mathbb{Z})$, Q_{4n} $(1 \leq n \in \mathbb{Z})$ and A_4 are given as follows:*

(D_0). $D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle$ *such that* $n \equiv 0 (mod 2)$:

(1) $\rho_{1,0} : a \mapsto 1, b \mapsto 1$; $\rho_{1,1} : a \mapsto 1, b \mapsto -1$; $\rho_{1,2} : a \mapsto -1, b \mapsto 1$; $\rho_{1,3} : a \mapsto -1, b \mapsto -1$;

(2) $\rho_{2,k} : a \mapsto \begin{pmatrix} \zeta_n^k & 0 \\ 0 & \zeta_n^{-k} \end{pmatrix}, b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *where k is an integer such that $1 \leq k \leq n/2 - 1$.*

(D_1). $D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle$ *such that* $n \equiv 1 (mod 2)$:

(1) $\rho_{1,0} : a \mapsto 1, b \mapsto 1$; $\rho_{1,1} : a \mapsto 1, b \mapsto -1$;

(2) $\rho_{2,k} : a \mapsto \begin{pmatrix} \zeta_n^k & 0 \\ 0 & \zeta_n^{-k} \end{pmatrix}, b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *where* k *is an integer such that* $1 \le k \le (n-1)/2$.

(Q_0). $Q_{4n} = \langle a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$ *such that* $n \equiv 0 (mod 2)$:

(1) $\rho_{1,0} : a \mapsto 1, b \mapsto 1$; $\rho_{1,1} : a \mapsto 1, b \mapsto -1$; $\rho_{1,2} : a \mapsto -1, b \mapsto 1$; $\rho_{1,3} : a \mapsto -1, b \mapsto -1$;

(2) $\rho_{2,l} : a \mapsto \begin{pmatrix} \zeta_{2n}^l & 0 \\ 0 & \zeta_{2n}^{-l} \end{pmatrix}, b \mapsto \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix}^l$, *where* l *is an integer such that* $1 \le k \le n-1$.

(Q_1). $Q_{4n} = \langle a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$ *such that* $n \equiv 1 (mod 2)$:

(1) $\rho_{1,0} : a \mapsto 1, b \mapsto 1$; $\rho_{1,1} : a \mapsto 1, b \mapsto -1$; $\rho_{1,2} : a \mapsto -1, b \mapsto \zeta_4$; $\rho_{1,3} : a \mapsto -1, b \mapsto -\zeta_4$;

(2) $\rho_{2,l} : a \mapsto \begin{pmatrix} \zeta_{2n}^l & 0 \\ 0 & \zeta_{2n}^{-l} \end{pmatrix}, b \mapsto \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix}^l$, *where* l *is an integer such that* $1 \le k \le n-1$.

(A_4). $A_4 = \langle a, b \rangle (\subset S_4)$, *where* $a = (123)$ *and* $b = (12)(34)$:

(1) $\rho_{1,0} : a \mapsto 1, b \mapsto 1$; $\rho_{1,1} : a \mapsto \zeta_3, b \mapsto 1$; $\rho_{1,2} : a \mapsto \zeta_3^{-1}, b \mapsto 1$;

(2) $\rho_3 : a \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, b \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. □

Proof. It is easy to check that all the representaions in the list above are well-defined, irreducible representations. Calculating the characters for a and b, we also see in each case that representations in the list are not isomorphic to one another. Now using the well-known equality $|G| = \sum_{k=1}^{l} (\dim V_i)^2$, where (ρ_i, V_i) $(1 \le k \le l)$ are all the non-isomorphic irreducible representations (eg. [Kt, Chap.8, page 270]), we find in each case that there are no other irreducible representations. □

Lemma (1.11). *Let* G *be a pre-C.Y. group and* A *a target abelian threefold. Assume that* G *is isomorphic to either* D_8, Q_8 *or* Q_{12}. *Then, under the notation in (1.10), the irreducible decomposition of the natural representation* $\rho : G \to SL(H^0(A, \Omega_A^1))$ *is of the form* $\rho = \rho_{1,1} \oplus \rho_{2,1}$ *if* $G \simeq D_8$ *and* $\rho = \rho_{1,0} \oplus \rho_{2,1}$ *if* $G \simeq Q_8$ *or* Q_{12}. *In particular, if* G *is a C.Y. group, then* $G \simeq D_8$ *and the action of* G *on* $H^0(A, \Omega_A^1)$ *is given as in (1.2)(3).*

Proof. Note that ρ is not isomorphic to a direct sum of three 1-dimensional representations, because G is non-commutative while ρ is injective (1.4)(1). Now using list in (1.10) together with (1.4)(1),(2), we obtain the desired irreducible decompositions. Using these decompositions, we readily find that $H^0(A, \Omega_A^1)^G = \{0\}$ if $G \simeq D_8$ and that $H^0(A, \Omega_A^1)^G \simeq \mathbb{C}$ if $G \simeq Q_8$ or Q_{12}. For the last statement in (1.11), we may just note that the representation $\rho_{2,1}$ of D_8 in (1.10) is equivalent to the one given by $a \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $b \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. □

The next two Lemmas are crucial and their proofs require geometric argument.

Lemma (1.12). *Neither $D_6(\simeq S_3)$ nor D_{12} is a pre-C.Y. group.*

Proof. Since D_6 can be embedded in D_{12}, the assertion for D_{12} follows from the one for D_6 (1.3). Assume the contrary that $D_6 = \langle a, b | a^3 = b^2 = 1, bab = a^{-1}\rangle$ is a pre-C.Y. group. Let A a target abelian threefold and $\rho : D_6 \to SL(H^0(A, \Omega^1_A))$ the natural representation. Then the same argument as in (1.11) shows that $\rho = \rho_{1,1} \oplus \rho_{2,1}$. Thus, there exists an appropriate basis $\langle v_1, v_2, v_3\rangle$ of $H^0(A, \Omega^1_A)$ under which $a^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^{-1} \end{pmatrix}$ and $b^* = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Let us fix an origin 0 of A and regard A as a group variety. Set $\alpha := a(0)$ and $\beta := b(0)$. Then, we have $a = t_\alpha \circ a_0$ and $b = t_\beta \circ b_0$ where $a_0, b_0 \in \mathrm{Aut}_{\mathrm{group}}(A)$. Set $\tilde{E} := \mathrm{Ker}(a_0 - id_A : A \to A)$. This is a subgroup scheme of A and is of dimension 1 by the description of a^*. Let us take the connected component E of $\tilde{E}$ containing the origin 0. Then E is an elliptic curve. Consider the quotient group $S := A/E$ and denote by π the natural projection $A \to S$. Since E is a one dimensional, S is an abelian surface.

Claim. *G descends to an automorphism group of S, that is, there exist automorphisms $\overline{a}$ and $\overline{b}$ of S such that $\overline{a} \circ \pi = \pi \circ a$ and that $\overline{b} \circ \pi = \pi \circ b$.*

Proof of Claim. Let F be a fiber of π. Then, there exists $s \in A$ such that $F = E+s$. Let x be a point of E. Then $a(x + s) = t_\alpha(a_0(e + s)) = t_\alpha(a_0(e) + a_0(s)) = a_0(e) + (a_0(s) + \alpha)$. Combining this with $a_0(E) = E$, we have $a(E + s) = E + (a_0(s) + \alpha)$. Thus, a descends to an algebraic automorphism $\overline{a}$ of S which maps $\pi(s)$ to $\pi(a_0(s)+\alpha)$. Similarly, we calculate $b(e+s) = b_0(e)+(b_0(s)+\beta)$. Note that $a_0b_0(0) = b_0a_0^{-1}(0) = 0$. Combining this with the equalities $ab = ba^{-1}$, $a^* = a_0^*$ and $b^* = b_0^*$, we get $a_0b_0 = b_0a_0^{-1}$. Thus, $a_0(b_0(e)) = b_0(a_0^{-1}(e)) = b_0(e)$, whence $b_0(e) \in \tilde{E}$ for $e \in E$. This together with $b_0(0) = 0 \in E$ implies that $b_0(e) \in E$. Thus $b(E + s) = E + (b_0(s) + \beta)$, whence, b also descends to an algebraic automorphism $\overline{b}$ of S. □

By construction, there exists a basis $\langle \overline{v_2}, \overline{v_3}\rangle$ of $H^0(S, \Omega^1_S)$ such that $\pi^*(\overline{v_2}) = v_2$ and $\pi^*(\overline{v_3}) = v_3$. Under this basis, the actions of $\overline{a}^*$ and $\overline{b}^*$ on $H^0(S, \Omega^1_S)$ are written as $\overline{a}^* = \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}$ and $\overline{b}^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. From this description, we see that $S^{\overline{a}}$ consists of isolated points. Now, using the canonical graded isomorphism $H^*(S, \mathbb{C}) = \oplus_{k=0}^4 \wedge^k (H^0(S, \Omega^1_S) \oplus \overline{H^0(S, \Omega^1_S)})$ and applying topological Lefschetz formula, we readily calculate that $|S^{\overline{a}}| = \sum_{k=0}^4 (-1)^k \mathrm{tr}(\overline{a}^* | H^k(S, \mathbb{C})) = 9$. On the other hand, the equality $ab = ba^{-1}$ gives an equality $\overline{a}\overline{b} = \overline{b}\overline{a}^{-1}$. Then $\overline{a}(\overline{b}(\overline{s})) = \overline{b}(\overline{a}^{-1}(\overline{s})) = \overline{b}(\overline{s})$ for $\overline{s} \in S^{\overline{a}}$. This implies that $\overline{b}$ acts on the nine point set $S^{\overline{a}}$. Combining this with the fact that $\overline{b}$ is of order 2, we find a point $\overline{s} \in S^{\overline{a}}$ such that $\overline{b}(\overline{s}) = \overline{s}$. Let F be the fiber of π over $\overline{s}$. Then, $b(F) = F$ and $b^*|H^0(F, \Omega^1_F) = -1$. Since F is an elliptic curve, F^b is then non-empty. However, this contradicts $A^{[G]} = \emptyset$. □

Lemma (1.13). *The group A_4 is not a pre-C.Y. group.*

Proof. Assume the contrary that $A_4 = \langle a, b\rangle$ is a pre-C.Y. group, where a and b are same as those in (1.10). Let A be a target abelian threefold and set $A = \mathbb{C}^3/\Lambda$, where Λ is a discrete sublattice of $\mathbb{C}^3$ of rank 6. (In this proof, we regard A as a three dimensional complex torus rather than an abelian variety.) Then, by the same argument as in (1.11), we readily find a basis $\langle v_1, v_2, v_3\rangle$ of $H^0(A, \Omega_A^1)$ under which $a^*|H^0(A,\Omega_A^1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $b^*|H^0(A,\Omega_A^1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Let us choose the global coordinates (z_1, z_2, z_3) of $\mathbb{C}^3$ such that $v_i = dz_i$ for all $i = 1, 2, 3$. Then there exist complex numbers α_i, β_i such that the (co-)actions of a and b on A are written as $a(z_1, z_2, z_3) = (z_2, z_3, z_1)+(\alpha_1, \alpha_2, \alpha_3)$ and $b(z_1, z_2, z_3) = (z_1, -z_2, -z_3)+ (\beta_1, \beta_2, \beta_3)$. Then, we readily calculate that $a^3(z_1, z_2, z_3) = (z_1, z_2, z_3) + (\alpha, \alpha, \alpha)$, where $\alpha := \alpha_1 + \alpha_2 + \alpha_3$. Since $a^3 = id$, we have $(\alpha, \alpha, \alpha) \in \Lambda$. Set $t := t_\alpha$. Then, we calculate that $b^{-1} \circ t \circ b(z_1, z_2, z_3) = (z_1, z_2, z_3) + (\alpha, -\alpha, -\alpha)$. On the other hand, since $\alpha \in \Lambda$, we have $b^{-1} \circ t \circ b = id$. Thus, $(\alpha, -\alpha, -\alpha) \in \Lambda$. In particular, $(2\alpha, 0, 0) = (\alpha, \alpha, \alpha) + (\alpha, -\alpha, -\alpha) \in \Lambda$. However, this implies $a^2(0, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3) = (2\alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3) = (0, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3) + (2\alpha, 0, 0)$, whence $a^2(P) = P$ for $P = [(0, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3)] \in A$, a contradiction. □

In order to finish the proof of (1.2), it only remains to show the following:

Lemma (1.14). *Let G be a group of order 24. Then G is not a C.Y. group.*

Proof. Assuming the contrary that there exists a C.Y. group G of order 24, and denoting its target abelain threefold by A, we shall derive a contradiction. For this, we make use of (1.3) and the following:

Proposition (1.15) (eg. [Bu, Chapter 9, Pages 171-174]). *Let G be an (arbitrary) group of order 24 and H_2 a 2−Sylow subgroup of G. Then H_2 is isomorphic to either C_8, $C_2 \oplus C_4$, $C_2^{\oplus 3}$, D_8 or Q_8 and G is isomorphic to one of the following 15 groups according to the isomorphism class of H_2:*

(I) $H_2 = \langle a\rangle \simeq C_8$:
- *(I_1) $G \simeq C_3 \times C_8$;*
- *(I_2) $G = \langle c, b\rangle \simeq C_3 \rtimes C_8$, where $a^{-1}ca = c^{-1}$.*

(II) $H_2 = \langle a, b\rangle \simeq C_2 \oplus C_4$:
- *(II_1) $G \simeq C_3 \times (C_2 \oplus C_4)$;*
- *(II_2) $G = \langle c, a, b\rangle \simeq C_3 \rtimes (C_2 \oplus C_4)$, where $a^{-1}ca = c$ and $b^{-1}cb = c^{-1}$.*
- *(II_3) $G = \langle c, a, b\rangle \simeq C_3 \rtimes (C_2 \oplus C_4)$, where $a^{-1}ca = c^{-1}$ and $b^{-1}cb = c$.*

(III) $H_2 = \langle a_1, a_2, a_3\rangle \simeq C_2^{\oplus 3}$:
- *(III_1) $G \simeq C_3 \times C_2^{\oplus 3}$;*
- *(III_2) $G = \langle a_1, a_2, a_3, c\rangle \simeq C_2^{\oplus 3} \rtimes C_3$, where $c^{-1}a_1c = a_1$, $c^{-1}a_2c = a_3$ and $c^{-1}a_3c = a_2a_3$;*
- *(III_3) $G = \langle c, a, b\rangle \simeq C_3 \rtimes C_2^{\oplus 3}$, where $a_1^{-1}ca_1 = c$, $a_2^{-1}ca_2 = c$ and $a_2^{-1}ca_2 = c^{-1}$.*

(IV) $H_2 = \langle a, b | a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1}\rangle \simeq Q_8$:
- *(IV_1) $G \simeq C_3 \times Q_8$;*
- *(IV_2) $G = \langle a, b, c\rangle \simeq Q_8 \rtimes C_3$, where $c^{-1}ac = b$, $c^{-1}bc = ab$;*
- *(IV_3) $G = \langle c, a, b\rangle \simeq C_3 \rtimes Q_8$, where $a^{-1}ca = c$, $b^{-1}cb = c^{-1}$.*

(V) $H_2 = \langle a, b | a^4 = 1, b^2 = 1, bab = a^{-1}\rangle \simeq D_8$:
- *(V_1) $G \simeq C_3 \times D_8$;*

(V_2) *$G = \langle c, a, b\rangle \simeq C_3 \rtimes D_8$, where $a^{-1}ca = c$, $b^{-1}cb = c^{-1}$;*
(V_3) *$G = \langle c, a, b\rangle \simeq C_3 \rtimes D_8$, where $a^{-1}ca = c^{-1}$, $b^{-1}cb = c$;*
(V_4) *$G \simeq S_4$.* □

In the case where $(I), (II), (III)$, H_2 is then a commutative pre-C.Y. group of order 8. However, this contradicts (1.5). In the case where (IV_1), (IV_3), (V_1), and (V_2), the subgroup $\langle a, c\rangle$ of G is isomorphic to C_{12}. However, this again contradicts (1.5). In the case where (V_4), G contains a subgroup which is isomorphic to A_4. However, this contradicts (1.13). Let us consider the case (IV_2). Set $H := \langle a, b\rangle$. By (1.11), the representation ρ_H of H on $H^0(A, \Omega^1_A)$ is decomposed as $\rho_H = \rho_{1,0} \oplus \rho_{2,1}$. Let us write the subspace of $H^0(A, \Omega^1_A)$ corresponding to $\rho_{1,0}$ by V_1. Since $ac = cb$, we have $a^*(c^*(x)) = c^*(b^*(x)) = c^*(x)$ for $x \in V_1$, whence V_1 is also G-stable. Thus, by the Maschke Theorem (eg. [Kt, Chap.8, Theorem (8.1)]), there exists a 2-dimensional G-stable subspace V_2 of $H^0(A, \Omega^1_A)$ such that $H^0(A, \Omega^1_A) = V_1 \oplus V_2$. Then under an appropriate basis of V_1 and V_2, the matrix representation of G on $H^0(A, \Omega^1_A)$ is of the form; $a^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_4 & 0 \\ 0 & 0 & -\zeta_4 \end{pmatrix}$, $b^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta_4 \\ 0 & \zeta_4 & 0 \end{pmatrix}$ and $c^* = \begin{pmatrix} \alpha & 0 \\ 0 & C \end{pmatrix}$, where α is a complex number and C is a 2×2 matrix. Since c is of order 3, α is either $1, \zeta_3$ or ζ_3^{-1}. If $\alpha = 1$, then $H^0(A, \Omega^1_A)^G = V_1 \neq 0$. However, this contradicts our assumption that G is a C.Y.group. Thus, we may assume that $\alpha = \zeta_3$ by replacing c by c^{-1} if necessary. Then, the eigen values of C are 1 and ζ_3^{-1}. However, then $(a^2)^*c^* = \begin{pmatrix} \zeta_3 & 0 \\ 0 & -C \end{pmatrix}$, whence the element a^2c does not have an eigen value 1 and this contradicts (1.4)(3). Hence the group in (IV_2) is not a C.Y. group. Finally, we consider the case (V_3). Set $V_1 := H^0(A, \Omega^1_A)^c$. Then, by (1.4)(3), $\dim V_1 = 1$. Using $ca = ac^{-1}$ and $cb = bc$, we see that V_1 is also stable under the actions a^* and b^*, whence G-stable. Then, again, by the Maschke Theorem, there exists a two-dimensional G-stable subspace V_2 of $H^0(A, \Omega^1_A)$ such that $H^0(A, \Omega^1_A) = V_1 \oplus V_2$. Note by (1.11) that this decomposition is also the irreducible decomposition of the representation of $\langle a, b\rangle (\simeq D_8)$. Thus, by (1.11), there exist a basis of V_1 and V_2 under which $a^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_4 & 0 \\ 0 & 0 & -\zeta_4 \end{pmatrix}$, $b^* = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $c^* = \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}$, where C is a 2×2 matrix. Then b^*c^* is of the form $\begin{pmatrix} -1 & 0 \\ 0 & D \end{pmatrix}$. Thus $\mathrm{ord}(bc) = 2$ by (1.4)(3). On the other hand, using $bc = cb$, $\mathrm{ord}(b) = 2$ and $\mathrm{ord}(c) = 3$, we see that $\mathrm{ord}(bc) = 6$, a contradiction. Hence the group in (V_3) is not a C.Y. group. Now we are done. □

§2. Calabi-Yau threefolds of Type K.

In this section, we study Calabi-Yau threefolds of Type K. As in Section 1, we define:

Definition (2.1). *We call a finite group G a Calabi-Yau group of Type K if there* *exist a K3 surface S, an elliptic curve E and a faithful representation $G \hookrightarrow Aut(S \times E)$ which satisfies the following conditions (1) - (4):*

(1) *G contains no elements of the form (id_S, non-zero translation of$E$);*
(2) *$g^*\omega_{S\times E} = \omega_{S\times E}$ for all $g \in G$;*
(3) *$(S \times E)^{[G]} = \emptyset$;*
(4) *$H^0(S \times E, \Omega^1_{S\times E})^G = \{0\}$.* $\square$

Throughout this section, we again abbreviate a Calabi-Yau group of Type K simply by a C.Y. group. The goal of this section is to prove the following:

Theorem (2.2). *Let G be a C.Y. group and $S \times E$ a target threefold. Then.*

(1) *G is isomorphic to either $C_2^{\oplus n}$ $(1 \le n \le 3)$, D_{2n} $(3 \le n \le 6)$ or $C_3^{\oplus 2} \rtimes C_2$;*
(2) *Conversely, each of these groups except possibly for D_{2n} $(3 \le n \le 6)$ and $C_3^{\oplus 2} \rtimes C_2$ is a C.Y. group;*
(3) *In each case of (1), the dimension $d(G)$ of the invariant part $H^2(S \times E)^G$ is calculated as in the following table:*

G	C_2	$C_2^{\oplus 2}$	$C_2^{\oplus 3}$	D_6	D_8	D_{10}	D_{12}	$C_3^{\oplus 2} \rtimes C_2$
$d(G)$	11	7	5	5	4	3	3	3

$\square$

As in section 1, we observe first that Theorem (2.2) implies the main Theorem [2].

Proof of the main Theorem [2] assuming Theorem (2.2).

First recall the following:

Lemma (2.3) [Be2, Page 8, Proposition]. *Let S be a K3 surface and E an elliptic curve. Then $Aut(S \times E) = Aut(S) \times Aut(E)$, that is, each element g of $Aut(S \times E)$ is of the form (g_S, g_E) where $g_S \in Aut(S)$ and $g_E \in Aut(E)$.*

Proof. It is sufficient to show that $\mathrm{Aut}(S \times E) \subset \mathrm{Aut}(S) \times \mathrm{Aut}(E)$. Let g be an element of $\mathrm{Aut}(S \times E)$. Note that the second projection $p_2 : S \times E \to E$ is nothing but the albanese morphism of $S \times E$. So, by the universality of albanese morphism, there exists an element g_E of E such that $p_2 \circ g = g_E \circ p_2$. In other words, g is of the form; $g : (s, e) \mapsto (g_S(s, e), g_E)$ and $e \in E \mapsto g_S(*, e)$ is then regarded as a morphism from E to $\mathrm{Aut}(S)$. However, since $H^0(S, T_S) = 0$ and $\mathrm{Aut}(S)$ is then descrete, this morphism must be constant, that is, $g_S(s, e)$ does not depend on $e \in E$. Thus, g is of the form, $g = (g_S, g_E)$ where $g_S \in \mathrm{Aut}(S)$ and $g_E \in \mathrm{Aut}(E)$. $\square$

Let X be a Calabi-Yau threefold of Type K. Fix a K3 surface S, an elliptic curve E and an etale Galois covering $\pi : S \times E \to X$ and denote by G the Galois group of π. Choose an origin of E and set $H := G \cap (\{id_S\} \times E)$. Then H is a normal subgroup of G by (2.3). Moreover, the induced action of G/H on the quotient threefold $(S \times E)/H = S \times (E/H)$ satisfies the conditions (1)-(4) in (1.1) and keeps the property $(S \times (E/H))/(G/H) = X$. So, replacing $(G, S \times E)$ by $(G/H, S \times (E/H))$, we may assume from the first that G itself is a C.Y. group and $S \times E$ its target threefold. Conversely, if G is a C.Y. group and $S \times E$ is its target product threefold, then $(S \times E)/G$ is a Calabi-Yau threefold of Type K. Therefore (2.2)(1) and (2) imply the main Theorem [2](1) and (2) respectively. The verification of [2](3) is exactly same as the one for C.Y. group of Type A. $\square$

Next, we observe that $C_2^{\oplus n}$ $(1 \le n \le 3)$ are actually C.Y. groups.

Proof of the fact that $C_2^{\oplus n}$ $(1 \le n \le 3)$ are C.Y. groups of Type K.

It is enough to find a K3 surface S and an elliptic curve E such that $C_2^{\oplus n}$ $(1 \le n \le 3)$ act on the product $S \times E$ as C.Y. groups. The following construction is much inspired by the work of Kondo [Ks]. Let us first take three elliptic curves with fixed origin E_1, E_2 and E and set $S := \mathrm{Km}(E_1 \times E_2)$, the smooth Kummer surface associated with the product abelian surface $E_1 \times E_2$. Fix elements $a_i, b_i \in (E_i)_2 - \{0\}$ for $i = 1, 2$ such that $a_i \neq b_i$. Then the three automorphisms of $E_1 \times E_2$ defined respectively by $(z_1, z_2) \mapsto (-z_1, -z_2)$, $(z_1, z_2) \mapsto (-z_1 + a_1, -z_2 + a_2)$, $(z_1, z_2) \mapsto (z_1 + b_1, z_2)$, and by $(z_1, z_2) \mapsto (z_1, z_2 + b_2)$ descend to those of $\mathrm{Aut}(S)$, which we denote by θ, t_1 and t_2 respectively. Let us fix $P_1, P_2 \in (E)_2 - \{0\}$ such that $P_1 \neq P_2$ and consider the three automorphisms of $S \times E$ defined by $\tilde{\theta} := (\theta, -1_E)$, $\tilde{t_1} := (t_1, t_{P_1})$ and $\tilde{t_2} := (t_2, t_{P_2})$. Then $G_1 := \langle \tilde{\theta} \rangle \simeq C_2$, $G_2 := \langle \tilde{\theta}, \tilde{t_1} \rangle \simeq C_2^{\oplus 2}$ and $G_3 := \langle \tilde{\theta}, \tilde{t_1}, \tilde{t_2} \rangle \simeq C_2^{\oplus 3}$. By construction, it is clear that each of $G_n \hookrightarrow \mathrm{Aut}(S \times E)$ satisfies the conditions (1), (3) and (4) in (2.1). Moerover, by using the explicit description of G_n, we can readily check that the condition (2) in (2.1) is also satisfied for each of $G_n \hookrightarrow \mathrm{Aut}(S \times E)$. Now we are done. $\square$

The rest of this section is devoted to prove (2.2)(1) and (2.2)(3).

Proof of (2.2)(1). First we note the following:

Lemma (2.4). *Let S be a K3 surface and g an element of finite order of $Aut(S)$ such that $S^{[\langle g \rangle]} = \emptyset$. Then,*

(1) *$g = id$ if $g^*\omega_S = \omega_S$; and*
(2) *g is of order 2 and $g^*\omega_S = -\omega_S$ if $g^*\omega_S \neq \omega_S$. Moreover, in this case, the quotient surface $S/\langle g \rangle$ is an Enriques surface.*

Proof. Set $n := \mathrm{ord}(g)$ and $T := S/\langle g \rangle$. Then T is a smooth projective surface such that $2 = \chi(\mathcal{O}_S) = n\chi(\mathcal{O}_T)$ and that $h^1(\mathcal{O}_T) = 0$. Thus, the pair $(n, \chi(\mathcal{O}_T))$ is either $(1, 2)$ or $(2, 1)$. Assume that $g^*\omega_S = \omega_S$. Then, ω_S descends to a nowhere vanishing holomorphic two form on T and T is then a K3 surface. Thus, $(n, \chi(\mathcal{O}_T)) = (1, 2)$. Assume next that $g^*\omega_S \neq \omega_S$. Then $(n, \chi(\mathcal{O}_T)) = (2, 1)$, whence g is an involution and satisfies $g^*\omega_S = -\omega_S$. The last part of (2) is nothing but (one of) the definition of an Enriques surface. $\square$

Lemma (2.5). *Let G be a C.Y. group, $S \times E$ its target threefold and $p_1 : G \to Aut(S)$ and $p_2 : G \to Aut(E)$ the natural projections under the identification $Aut(S \times E) = Aut(S) \times Aut(E)$ (2.3). Set $G_S := Im(p_1)$ and $G_E := Im(p_2)$. Then $G_S \simeq G \simeq G_E$ through p_1 and p_2.*

Proof. It is sufficient to show that both p_i are injective. Let g be an element of $\mathrm{Ker}(p_1)$. Then g is of the form $g = (id_S, g_E)$ such that $g_E^*\omega_E = \omega_E$

(2.1)(3). Then g_E is a translation of E whence $g_E = id_E$ by (2.1)(1). Hence p_1 is injective. Let g be an element of $\mathrm{Ker}(p_2)$. Then g is of the form $g = (g_S, id_E)$. By (2.1)(2)(3), g_S satisfies that $S^{[\langle g_S \rangle]} = \emptyset$ and that $g_S^* \omega_S = \omega_S$. Now, we deduce from (2.4) that $g_S = id_S$. Therefore p_2 is also injective. □

Lemma (2.6). *Let G be a C.Y. group and $S \times E$ its target threefold. Then, there exists a normal commutative subgroup H of G such that*

(1) *$H \neq G$ and if $\iota \in G - H$ then ι is of order 2 and $G = H \rtimes < \iota >$, where the semi-direct product structure is defined by $\iota h \iota = h^{-1}$ for h being in H; and*
(2) *there exist positive integers n and m such that $n|m$ and that $H \simeq C_n \oplus C_m$.*

Moreover, this subgroup satisfies that $h_S^* \omega_S = \omega_S$ if $h \in H$ and that $\iota_S^* \omega_S = -\omega_S$ and $S^{\iota_S} = \emptyset$ if $\iota \in G - H$.

Proof. Let H_E be the kernel of the natural representation $G_E \to \mathrm{GL}(H^0(E, \Omega_E^1))$. This implies that the corresponding subgroup H_S of G_S acts trivially on $H^0(S, \Omega_S^2)$. Since $H^0(S \times E, \Omega_{S\times E}^1)^H \simeq H^0(E, \Omega_E^1)^{H_E} \simeq \mathbb{C}$, we see that $H_E \neq G_E$. Let ι_E be an arbitrary element of $G_E - H_E$ and set $\iota := (\iota_S, \iota_E) \in \mathrm{Aut}(S \times E)$. Then there exists a complex number α such that $\alpha \neq 1$ and that $\iota_E^* \omega_E = \alpha \omega_E$. Note that $E^{\iota_E} \neq \emptyset$. Then $\iota_S \omega_S = \alpha^{-1} \omega_S$ and $S^{\iota_S} = \emptyset$. Therefore ι_S is an involution and $\alpha = -1$. Let us fix one of such $\iota \in G - H$ and choose another $\iota' \in G - H$. Then, $(\iota_E \circ \iota'_E)^* \omega_E = \omega_E$, whence $\iota'_E \circ \iota_E \in H_E$. Therefore, $G_E = H_E \rtimes \langle \iota_E \rangle$. Fix the origin 0 of E so as to being $0 \in E^{\iota_E}$. Then $\iota_E = -1_E$ and $-1_E \circ t_a \circ -1_E = t_{-a} = t_a^{-1}$. In particular, $\iota_E \circ h_E \circ \iota_E = h_E^{-1}$ if $h_E \in H_E$. Moreover, since H_E consists of translations of E, there exist positive integers n and m such that $H_E \simeq C_n \oplus C_m$ and that $n|m$. Now the result follows from (2.5). □

In order to finish the proof of (2.2)(1), it remains to show $(n, m) \in \{(1, k)(1 \leq k \leq 6), (2, 2), (3, 3)\}$. For this we make use of the following:

Theorem (2.7) [Ni, Page 106, Section 5, Paragraph 8]. *Let S be a K3 surface.*

(1) *Let $g \neq id$ be an element of $Aut(S)$ of finite order such that $g^* \omega_S = \omega_S$. Set $n := ord(g)$. Then $n \leq 8$. Moreover, S^g is a finite set and its cardinality $|S^g|$ is given as in the following table:*

ord(g)	2	3	4	5	6	7	8
$\lvert S^g \rvert$	8	6	4	4	2	3	2

(2) *Let H be a finite, commutative subgroup of $Aut(S)$. Assume that H is symplectic, that is, $g^* \omega_S = \omega_S$ for each $g \in H$. Then H is isomorphic to either one of C_n $(1 \leq n \leq 8)$, $C_2^{\oplus n}$ $(2 \leq n \leq 4)$, $C_2 \oplus C_4$, $C_2 \oplus C_6$, $C_3^{\oplus 2}$, or $C_4^{\oplus 2}$.* □

Due to (2.7) and the fact that H_S is a commutative symplectic automorphism group of S of the form $G_S \simeq C_n \oplus C_m$ (2.6), it is now sufficient to show that $(n, m) \neq (1, 7), (1, 8), (2, 4), (2, 6), (4, 4)$.

Assume that $(n, m) = (1, 7)$. Then $H_S = \langle h_S \rangle \simeq C_7$ and $G_S = \langle h_S, \iota | \iota \circ h_S \circ \iota = h_S^{-1} \rangle$. Thus ι_S acts on S^{h_S}. On the other hand, since S^{h_S} consists of three points, $(S^{h_S})^{\iota_S} \neq \emptyset$, a contradiction to $S^{\iota_S} = \emptyset$.

Assume that $(n,m)=(1,8)$. Then $H_S=\langle h_S\rangle\simeq C_8$ and $G_S=\langle h_S,\iota|\iota\circ h_S\circ\iota=h_S^{-1}\rangle$. Thus ι_S and h_S act on $S^{h_S^2}-S^{h_S}$. Note that $|S^{h_S^2}-S^{h_S}|=2$ by (2.7)(1). We set $S^{h_S^2}-S^{h_S}=\{P_1,P_2\}$. Since $S^{\iota_S}=S^{h_S}=\emptyset$, we have $\iota_S(P_1)=h_S(P_1)=P_2$. However, then $\iota_S\circ h_S\in G_S-H_S$ while $P_1\in S^{\iota_S\circ h_S}=\emptyset$, a contradiction.
Assume that $(n,m)=(2,4)$. Then $H_S=\langle g_S\rangle\oplus\langle h_S\rangle\simeq C_2\oplus C_4$. As before, $\langle g_S,h_S,\iota_S\rangle/\langle h_S^2\rangle\simeq C_2^{\oplus 3}$ acts on the set $S^{h_S^2}-S^{h_S}$ consisting of four points by (2.7)(1). Thus, we have a natural representation $\varphi: C_2^{\oplus 3}\to S_4$ through this action. Since S_4 does not admits a subgroup isomorphic to $C_2^{\oplus 3}$, $\mathrm{Ker}(\varphi)\neq\{id\}$. Moreover, since $S^f=\emptyset$ if $f\in G_S-H_S$, $\mathrm{Ker}(\varphi)\subset\langle g_S,h_S\rangle/\langle h_S\rangle$. Let $\alpha\in\langle g_S,h_S\rangle$ is a lift of a non-trivial element of $\mathrm{Ker}(\varphi)$. Since $h_S\notin\mathrm{Ker}(\varphi)$, $\langle\alpha,h_S^2\rangle$ is isomorphic to either $C_2^{\oplus 2}$ or $C_2\oplus C_4$. On the other hand, letting $P\in S^{h_S^2}-S^{h_S}$, we have a natural injection $\langle\alpha,h_S^2\rangle\hookrightarrow\mathrm{SL}(T_{S,P})\simeq\mathrm{SL}(2,\mathbb{C})$. However, this contradicts the following well-known:

Theorem (2,8) eg. [Su, Chap.3 section 6, Theorem 6.17]. *Let G be a finite subgroup of $SL(2,\mathbb{C})$. Then G is isomorphic to either one of C_n, Q_{4n}, T_{24}, O_{48} or I_{120}.* □

Assume that $(n,m)=(2,6)$. Then $H_S=\langle g_S\rangle\oplus\langle h_S\rangle\simeq C_2\oplus C_6$. Then S^{h_S} consists of 2 points. Set $S^{h_S}=\{P_1,P_2\}$. As before, $\langle g_S,\iota_S\rangle$ acts on $\{P_1,P_2\}$ and satisfies $\iota_S(P_1)=P_2$ and $\iota_S(P_2)=P_1$. Assume that $g_S(P_1)=P_1$. Then $\langle g_S,h_S\rangle\simeq C_2\oplus C_6\hookrightarrow\mathrm{SL}(T_{S,P_1})\simeq\mathrm{SL}(2,\mathbb{C})$, a contradiction to (2.8). Assume that $g_S(P_1)=P_2$. Then, $\iota\circ g_S(P_1)=P_1$, a contradiction to $\iota\circ g_S\in G_S-H_S$.
Assume that $(n,m)=(4,4)$. Then $H_E=\langle g_S\rangle\oplus\langle h_S\rangle\simeq C_4\oplus C_4$. Set $S^{g_S}=\{P_1,P_2,P_3,P_4\}$. Then $\langle\iota_S,h_S\rangle$ acts on S^{g_S}. Since neither $\langle g_S,h_S\rangle\simeq C_4^{\oplus 2}$ nor $\langle g_S,h_S^2\rangle\simeq C_4\oplus C_2$ can be embeded into $\mathrm{SL}(2,\mathbb{C})$ by (2.8), after renumbering if necessary, we have $h_S^i(P_1)=P_{i+1}$ for $1\le i\le 3$. Set $\iota_S(P_1)=P_{i+1}$. Then $1\le i\le 3$ and $\iota_S\circ h_S^i(P_1)=P_1$, a contradiction to $\iota\circ h_S^i\in G_S-H_S$. Now we are done. Q.E.D. for (2.2)(1). □

Proof of (2.2)(3). Using the Kunneth formula and (2.5), we have $H^2(S\times E,\mathbb{C})^G\simeq H^2(S,\mathbb{C})^{G_S}\otimes H^0(E,\mathbb{C})\oplus H^0(S,\mathbb{C})\otimes H^2(E,\mathbb{C})\simeq H^2(S,\mathbb{C})^{G_S}\oplus\mathbb{C}$. Threfore, it is sufficient to calculate $\dim H^2(S,\mathbb{C})^{G_S}$. We carry out this calculation by dividing into cases according to the isomorphism classes of G. In what follows, we demonstrate how to calculate $\dim H^2(S,\mathbb{C})^{G_S}$ only for the most typical case $G:=\langle a\rangle\rtimes\langle b\rangle\simeq C_6\rtimes C_2=D_{12}$ (and the calculation for other cases, which is similar to the one of this case, is left to the readers as an exercise). From now on, for simplicity, we denote G_S by G. Under the notation in (1.10), the irreducible decomposition of the natural representation of G on $H^2(S,\mathbb{C})$ is written as

$$H^2(S,\mathbb{C})=\rho_{1,0}^{\oplus p}\oplus\rho_{1,1}^{\oplus q}\oplus\rho_{1,2}^{\oplus r}\oplus\rho_{1,3}^{\oplus s}\oplus\rho_{2,1}^{\oplus t}\oplus\rho_{2,2}^{\oplus u}-(1).$$

Let us determine the values p,q,r,s,t,u by applying the topological Lefschetz fixed point formula:

$$\chi_{top}(S^g)=\sum_{k=0}^{4}(-1)^k\mathrm{tr}(g^*|H^k(S,\mathbb{C}))=2+\mathrm{tr}(g^*|H^2(S,\mathbb{C}))-(2).$$

Comparing the dimension of the both sides of (1), we have

$22 = p + q + r + s + 2t + 2u$ - (3).
Note that $|S^a| = 2$ by (2.7)(1). Then, combining (1) and (2) with $|S^a| = 2$, we have $2 = \chi_{top}(S^a) = 2 + p\mathrm{tr}(\rho_{1,0}(a)) + \cdots + u\mathrm{tr}(\rho_{2,2}(a)) = 2 + a + b - c - d + e - f$. This gives
$0 = p + q - r - s + t - u$ - (4).
Similarly, from $|S^{a^2}| = 6$, $|S^{a^3}| = 8$ and $S^b = S^{ab} = \emptyset$, we deduce
$4 = p + q + r + s - t - u$ - (5)
$6 = p + q - r - s - 2t + 2u$ - (6)
$-2 = p - q + r - s$ - (7) and
$-2 = p - q - r + s$ - (8).
Now solving the system of equations (3) - (8), we readily obtain
$p = 2$, $q = 4$, $r = 2$, $s = 2$, $t = 2$ and $u = 4$.
This implies $\dim H^2(S, \mathbb{C})^G = p = 2$. Therefore $\dim H^2(S \times E\mathbb{C})^G = 2 + 1 = 3$ in the case where $G \simeq D_{12}$. □

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